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Discrete opinion dynamics on networks based on social influence

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Abstract

A model of opinion dynamics based on social influence on networks was studied. The opinion of each agent can have integer values $i = 1, 2, \ldots, I$ and opinion exchanges are restricted to connected agents. It was found that for any $I \geqslant 2$ and self-confidence parameter $0 \leqslant u < 1$, when u is a degree-independent constant, the weighted proportion $\langle q_i \rangle$ of the population that hold a given opinion i is a martingale, and the fraction q_i of opinion i will gradually converge to $\langle q_i \rangle$. The tendency can slow down with the increase of degree assortativity of networks. When u is degree dependent, $\langle q_i \rangle$ does not possess the martingale property, however q_i still converges to it. In both cases for a finite network the states of all agents will finally reach consensus. Further if there exist stubborn persons in the population whose opinions do not change over time, it was found that for degree-independent constant u, both q_i and $\langle q_i \rangle$ will converge to fixed proportions which only depend on the distribution of initial obstinate persons, and naturally the final equilibrium state will be the coexistence of diverse opinions held by the stubborn people. The analytical results were verified by numerical simulations on Barabási–Albert (BA) networks. The model highlights the influence of high-degree agents on the final consensus or coexistence state and captures some realistic features of the diffusion of opinions in social networks.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

In recent years, there has been a lot of interest in the application of statistical physics paradigms for a quantitative description and comprehension of collective social behavior of individuals and economic processes (see [1] and references therein). An interesting research subject is opinion dynamics in socio-economic systems, which is a part of sociophysics [2, 3] and has become a main stream of research in physics [4–19].

An individual opinion can be defined by a finite number of integers as in the model proposed by Sznajd *et al* [7] and the social agents update their opinions as a result of social influence, often according to some version of a majority rule or imitation [10, 20, 21]. Most models on discrete opinion dynamics are based on binary opinions, e.g. two states—spin up and spin down. Clusters of opposite opinions appear when the dynamics occur on a social network with exchanges restricted to connected agents and these patterns resemble magnetic domains in Ising ferromagnets [22]. Opinions can also be represented by real numbers, i.e. the continuous opinions, as in the model proposed by Deffuant *et al* [23]. In both cases generally the dynamical processes have a natural absorbing state or consensus, in which all the agents share the same opinion [24]. Other models such as the one by Hegselmann and Krause [25], the one based on social impact theory [26], the voter model [10–16], Galam's majority rule model [27–29] and Axelrod's model [30] have been reviewed in [31]. As a practical application, some researchers have also recently applied opinion dynamics to study the spreading of technological innovations in a society [32].

In this paper, we study a social influence-based model of discrete opinion dynamics on networks with agents located on the nodes of the networks. To make the model more generic we assume, as is reasonable, that the initial number of opinions distributed in the whole society is $I \geqslant 2$, which is like a q-state Potts model [33, 34]. Besides for each agent we assign a self-confidence parameter $0 \leqslant u < 1$ quantifying the extent to which the agent believes his own choice, not subject to the opinions of his acquaintances. Instead of focusing on the mean time to reach consensus in popular literatures on opinion dynamics, we study the statistical properties of proportion q_i and weighted proportion $\langle q_i \rangle$ of opinion i by analytical calculations and computer simulations.

First we study the case that all agents are undecided ones. It was shown that for any $I \geqslant 2$, if $0 \leqslant u < 1$ is a constant independent of agent degrees, $\langle q_i \rangle$, which integrates the information of degrees of the people holding opinion i, is a martingale, and q_i will converge to $\langle q_i \rangle$. A more realistic scenario is that u is degree dependent, in which the convergence property still holds. Second we also study the case that in the population there are stubborn persons whose opinions never change. It was found that q_i will converge to a proportion which depends on both $\langle q_i \rangle$ and the initial fraction of obstinate agents. Specifically if u is a degree-independent constant, both $\langle q_i \rangle$ and q_i will converge to fixed constants which only depend on the initial distribution of decided agents.

2. Model

Our model is implemented on a social network which is a connected graph with a certain degree distribution. The nodes of the network represent agents in a society and edges represent their social connections. In the following discussion, for practical purposes the network structure is considered static over the time. We note that besides ordinary binary opinions or choices, such as PC–Macintosh, Windows–Linux, coffee–tea, etc, in real life there exist many situations in which the number of available choices is more than two, e.g. adopting the produce of some brand or supporting some political party in elections of multi-party countries. Thus we consider a general case and assume that the initial number of distinct opinions is $I \geqslant 2$. The opinion of each agent can take discrete values $i = 1, 2, \ldots, I$.

The model is described as follows. Starting from an initial opinion distribution, agents asynchronously update their opinions at a rate λ. Namely during any time interval $\mathrm{d}t$, each agent updates his opinion (makes a decision as to which opinion to hold) with probability

Table 1. The basic parameters in the model.

Parameter	Meaning
N	total number of nodes
n_k	number of nodes with degree k
$p_k = n_k/N$	degree distribution
$m_{i,k}$	number of nodes with degree k and opinion i
m_i	total number of nodes with opinion i
$q_i = m_i/N$	fraction of nodes with opinion i
$q_{i,k} = m_{i,k}/n_k$	fraction of nodes with opinion i in all nodes with degree k
u	probability of maintaining current opinion
$v = 1 - u$	probability of randomly selecting a neighbor's opinion

$\lambda\,\mathrm{d}t$, based on the opinions of his neighbors. Due to the uncertainties concerning something new, e.g. new brands of produce or political candidates, agents have to collect many opinions of their acquaintances before taking any decisions; however, we also believe that each agent has the capability of judgment and self-confidence to some extent, not completely subject to the opinions of his neighbors. Thus we propose a parameter u which can characterize the 'confidence' of the agents and weights how much the agent trusts his own opinion with respect to those of others. Specifically with probability $0 \leqslant u < 1$ a given agent maintains his current opinion, and with probability $v = 1 - u$ the agent randomly selects one of his neighbors and sets his new opinion to be the same as that neighbor, which is equivalent to assuming that the probability that his new opinion will be i is c/d (the psychology of conformity) in this case where c is the number of his neighbors with opinion i and d his degree.

The basic parameters and corresponding meanings for the model are listed in table 1. When $u = 0$ and $I = 2$, the model is reduced to the general voter model which has been studied extensively [13, 14, 16, 35]. In a finite system, the only possible steady state of the dynamical process is the fully ordered state, in which all agents have the same opinion.

For the pattern evolution of the dynamics model, a key factor is that agents interact and this generally tends to make neighborhood people more similar in opinion. On short timescales, coexisting ordered domains of small size are formed. Repeated interactions in time and space, i.e. a coarsening process of such domains [36, 37], lead to higher degree of homogeneity and larger clusters with identical opinions. The drive toward order is provided by the pressure of the majority of the agent's peers in an average sense and the tendency of interacting agents to become more alike. The effect is often termed 'social influence' [38] and can be seen as a counterpart of ferromagnetic interaction in magnets.

3. Evolution of opinions for undecided agents

In this section, we first suppose that $0 < v \leqslant 1$ is degree independent and for each agent the value of v is the same and study the situation in section 3.1. In section 3.2 we will discuss a more realistic case in which v is degree dependent, say it is a decreasing function of k. In the long term, since $v > 0$, every agent has the chance of randomly selecting a neighbor's opinion as his updated one, and thus we could call the agents undecided ones.

3.1. When v is degree independent

Because all N agents update their opinions asynchronously and independently of each other at the same rate, everyone has the same chance to be observed updating at time t. Thus the probability that the update changes a degree-k agent from opinion not i to i is

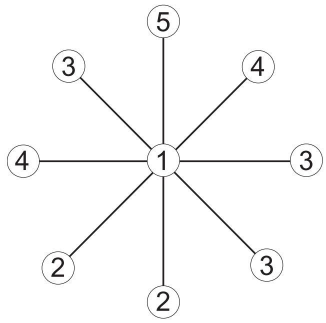

Figure 1. A star network with $N = 9$ agents and $I = 5$ opinions. The numbers in nodes represent corresponding opinions.

$$P_{\bar{i}\to i}(k) = p_k(1 - q_{i,k})v\frac{\sum_j jp_jq_{i,j}}{\sum_j jp_j} \tag{1}$$

and the probability that the update changes a degree-k agent from opinion i to not i is

$$P_{i\to\bar{i}}(k) = p_kq_{i,k}v\frac{\sum_j jp_j(1 - q_{i,j})}{\sum_j jp_j} = p_kq_{i,k}v\left(1 - \frac{\sum_j jp_jq_{i,j}}{\sum_j jp_j}\right). \tag{2}$$

Note that equations (1) and (2) are only valid for uncorrelated or weakly correlated networks.

We define

$$\langle q_i\rangle = \frac{\sum_j jp_jq_{i,j}}{\sum_j jp_j} = \frac{\sum_j jm_{i,j}}{N\langle k\rangle} \tag{3}$$

as the weighted fraction of opinion i ($\langle k\rangle$ is the mean degree) and it is the fraction of total degree of agents with opinion i in the total degree of the whole network. For the star network in figure 1, $q_1 = 1/9$ while $\langle q_1\rangle = 8/16$, which results from the large degree of agent with opinion 1. The definition can reveal, as shown in the following discussion, the influence of high-degree agents on the final state of the dynamics model.

Equations (1) and (2) can be rewritten as

$$\begin{cases} P_{\bar{i}\to i}(k) = p_k(1 - q_{i,k})v\langle q_i\rangle \\ P_{i\to\bar{i}}(k) = p_kq_{i,k}v(1 - \langle q_i\rangle). \end{cases} \tag{4}$$

A particular update yields the following increment of $m_{i,k}$:

$$\Delta m_{i,k} = \begin{cases} +1 & \text{with probability} \quad p_k(1 - q_{i,k})v\langle q_i\rangle \\ -1 & \text{with probability} \quad p_kq_{i,k}v\,(1 - \langle q_i\rangle) \\ \ \ 0 & \text{otherwise.} \end{cases} \tag{5}$$

For the whole network the updating process is a Poisson process of rate $N\lambda$. Therefore the increase of $m_{i,k}$ in an interval $(t, t + \mathrm{d}t)$ is

$$\Delta' m_{i,k} = \begin{cases} +1 & \text{with probability} \quad n_k(1 - q_{i,k})v\langle q_i\rangle\lambda\,\mathrm{d}t \\ -1 & \text{with probability} \quad n_kq_{i,k}v\,(1 - \langle q_i\rangle)\,\lambda\,\mathrm{d}t \\ \ \ 0 & \text{otherwise.} \end{cases} \tag{6}$$

The mean of $\Delta' m_{i,k}$ is

$$E(\Delta' m_{i,k}) = n_k(\langle q_i\rangle - q_{i,k})v\lambda\,\mathrm{d}t \tag{7}$$

and its second moment is given by

$$E[(\Delta' m_{i,k})^2] = n_k(1 - q_{i,k})v\langle q_i\rangle\lambda\,\mathrm{d}t + n_k q_{i,k} v(1 - \langle q_i\rangle)\lambda\,\mathrm{d}t$$
$$= n_k(\langle q_i\rangle + q_{i,k} - 2\langle q_i\rangle q_{i,k})v\lambda\,\mathrm{d}t. \qquad (8)$$

Thus the variance of $\Delta' m_{i,k}$ is

$$\mathrm{Var}(\Delta' m_{i,k}) = n_k\sigma_{i,k}^2 v\lambda\,\mathrm{d}t + o(\mathrm{d}t^2), \qquad (9)$$

where $\sigma_{i,k}^2 = \langle q_i\rangle + q_{i,k} - 2\langle q_i\rangle q_{i,k}$.

According to the definition of $q_{i,k}$, we have

$$E(\Delta q_{i,k}) = \frac{E(\Delta' m_{i,k})}{n_k} = (\langle q_i\rangle - q_{i,k})v\lambda\,\mathrm{d}t \qquad (10)$$

and

$$\mathrm{Var}(\Delta q_{i,k}) = \frac{\mathrm{Var}(\Delta' m_{i,k})}{n_k^2} \simeq \frac{1}{n_k}\sigma_{i,k}^2 v\lambda\,\mathrm{d}t. \qquad (11)$$

When the number of agents N is large, equations (10) and (11) can be approximately described by the following Langevin equation:

$$\mathrm{d}q_{i,k} = (\langle q_i\rangle - q_{i,k})v\lambda\,\mathrm{d}t + \frac{1}{\sqrt{n_k}}\sigma_{i,k}\sqrt{\lambda v}\,\mathrm{d}B_t, \qquad (12)$$

where B_t is the k-independent Brownian motion. We redefine the time unit so that $\lambda = 1$, thus

$$\mathrm{d}q_{i,k} = (\langle q_i\rangle - q_{i,k})v\,\mathrm{d}t + \frac{1}{\sqrt{n_k}}\sigma_{i,k}\sqrt{v}\,\mathrm{d}B_t. \qquad (13)$$

According to equation (3), we obtain

$$\mathrm{d}\langle q_i(t)\rangle = \frac{\sum_k kp_k\left[\frac{1}{\sqrt{n_k}}\sigma_{i,k}\sqrt{v}\,\mathrm{d}B_t\right]}{\sum_k kp_k}. \qquad (14)$$

Therefore $\langle q_i(t)\rangle$ is a martingale, and its mean

$$E[\langle q_i(t)\rangle] = \text{const}. \qquad (15)$$

When N is large $1/\sqrt{n_k}$ is small, thus we can neglect the fluctuation term in equation (13) and obtain

$$\frac{\mathrm{d}q_{i,k}}{\mathrm{d}t} = (\langle q_i\rangle - q_{i,k})v. \qquad (16)$$

We can divide the agents into different groups according to their degrees, so that all agents in the same group have the same degree. When N is large the size n_k of each group is also large, then we can approximately neglect the fluctuations within each group and replace the groupwise random variables $m_{i,k}, q_{i,k}$ by their mean values. In this manner, equation (16) can be regarded as a set of normal differential equations which contain deterministic variables only and its solution is

$$q_{i,k}(t) = q_{i,k}(0)\,\mathrm{e}^{-vt} + \langle q_i\rangle(1 - \mathrm{e}^{-vt}). \qquad (17)$$

Thus we have

$$\lim_{t\to\infty} q_{i,k}(t) = \langle q_i\rangle. \qquad (18)$$

According to $q_i(t) = \sum_k n_k q_{i,k} \big/ \sum_k n_k$, we obtain

$$q_i(t) = q_i(0)\,\mathrm{e}^{-vt} + \langle q_i\rangle(1 - \mathrm{e}^{-vt}). \qquad (19)$$

Thus

$$\lim_{t\to\infty} q_i(t) = \langle q_i\rangle \tag{20}$$

and

$$\lim_{t\to\infty} E[q_i(t)] = E[\langle q_i\rangle]. \tag{21}$$

The above-obtained results can apply to uncorrelated or weakly correlated networks with arbitrary degree distributions. From the definition of $\langle q_i\rangle$, we know that each agent is given a weight equal to his degree k. Thus equation (20) reveals that high-degree agents contribute more to the final fraction of opinions than the low-degree ones and thus they are more influential, which is consistent with the real-life condition that a relatively small number of people with high social status and prestige can affect a significant proportion of the whole society in their opinion shift.

An instructive example is the extreme case of the star network (see figure 1), where $N-1$ agents are connected only to a single central hub. For our model, if the hub is in state 1 and all other agents are in other states ($I = 5$), then equations (3) and (15) predict that $E[\langle q_1(t)\rangle] = \langle q_1(0)\rangle = 1/2$, though the proportion of agents with opinion 1 is only $1/N$. The conclusion agrees with previous analytical results for $I = 2$ [14]. Thus a single individual with a considerable number of neighbors largely determines the final state.

Generally the initial q_i is not equal to $\langle q_i\rangle$. If we distribute each opinion i randomly on agents in the network, the initial q_i will be approximately equal to $\langle q_i\rangle$. However for heterogeneous networks, such as scale-free ones, if we constrainedly assign a specific opinion i to the hub agents with the largest degrees, the initial $\langle q_i\rangle$ will be much larger than q_i.

We know that different social networks can have different degree distributions. Many real-life social networks are exponential, while the degree distribution of some particular social networks, such as sexual contacts, is a power law [39]. The degree distribution of many scientific collaboration networks is usually fat-tailed [40]. Recent research on online social networks reveals that the degree distribution of these social networks is commonly a power law [41–44]. Many researchers have used power-law networks or a BA network as the model of social networks [45–47]. It is plausible to assume that social networks have a scale-free network structure. Thus we will validate the above conclusions by numerical simulations performed on a BA network of size $N = 10^4$ and mean degree $\langle k\rangle = 6$ ($m = 3$) [48]. The degree distribution of the BA network is a power-law $p_k \sim k^{-3}$. The initial $I = 5$ opinions are randomly assigned. We then randomly select one agent in the network and with probability v update his opinion to be the opinion of one of his neighbors who is picked randomly. These steps constitute the sample paths of q_i and $\langle q_i\rangle$ and we record the data after every N/v steps. We repeat this experiment 100 times, so that 100 sample paths are collected for each q_i and $\langle q_i\rangle$. We then take the mean of q_i and $\langle q_i\rangle$ over the sample paths and obtain the estimates for $E(q_i)$ and $E[\langle q_i\rangle]$.

In this case the average values of q_i are very close to those of $\langle q_i\rangle$, thus for distinguishability we give the average values of $\langle q_i\rangle$ (figure 2(*a*)) and the differences between the mean values of $\langle q_i\rangle$ and q_i (figure 2(*b*)). It is clear that for random assignment of opinions the initial q_i is approximately equal to $\langle q_i\rangle$. Besides figure 2 also confirms equation (21), i.e. the convergence of $E(q_i)$ to $E[\langle q_i\rangle]$, and the martingale property of $\langle q_i\rangle$, i.e. $E[\langle q_i\rangle]$ does not change with time. Figure 2(*c*) shows the error bars with ± 1 standard deviation for q_i and $\langle q_i\rangle$. Since all agents are undecided, the standard deviations of q_i and $\langle q_i\rangle$ increase with T, which leads to indistinguishability of error bars between the two lines with close mean values, thus we show error bars only for q_1 and $\langle q_5\rangle$.

To show that a small number of agents with large degrees can significantly affect the final state of the evolution of opinions, we constrainedly assign the opinion 5 to the 100 largest

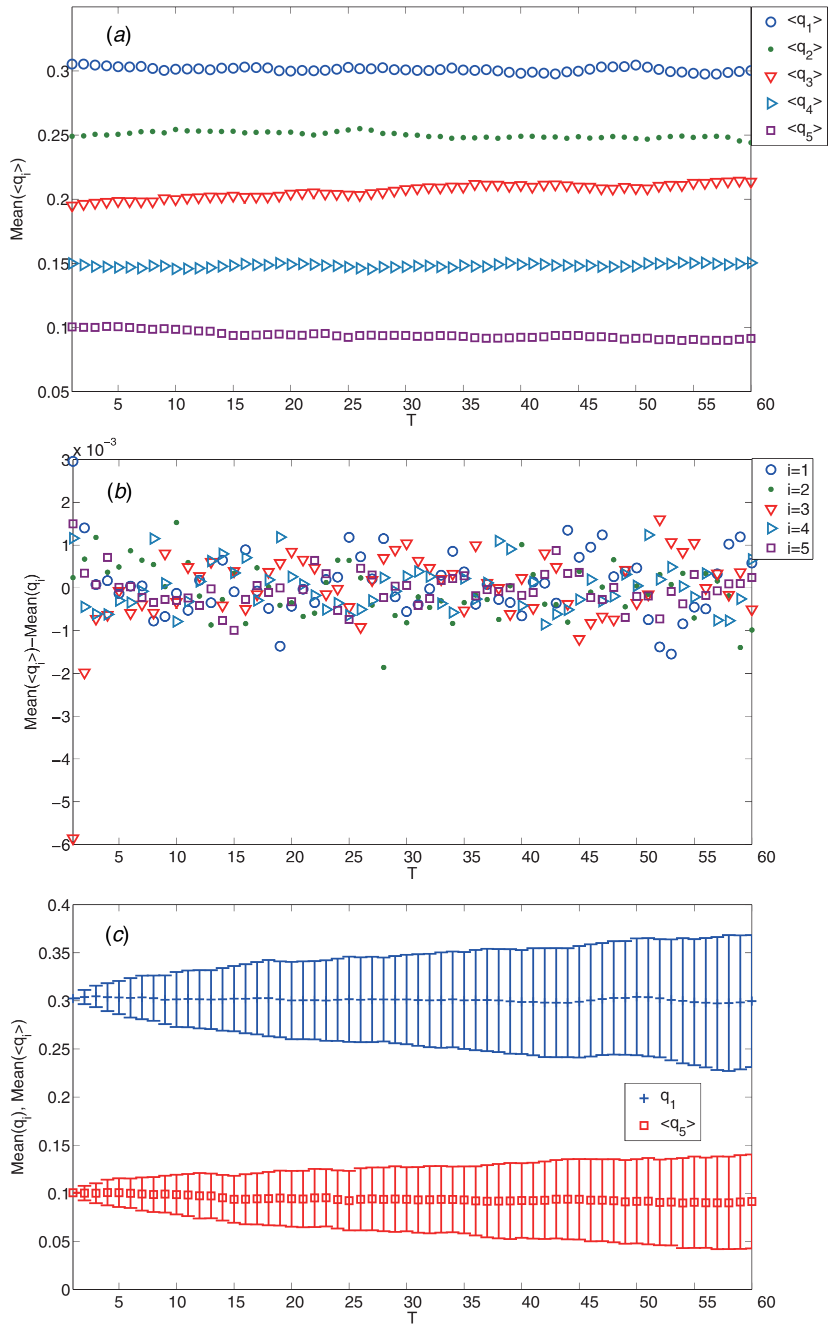

Figure 2. Evolutions of mean values of q_i and $\langle q_i \rangle$ for $v = 0.7$. The unit of time T is N/v steps. The simulation is performed on a BA network with 10^4 agents and mean degree 6.

degree agents of the BA network. As shown in figure 3(a), the large-degree agents contribute a considerable weight to $\langle q_5 \rangle$ and its initial value is much larger than q_5. It is also found that $\langle q_i \rangle$ is still a martingale, and the mean values of q_i approach mean values of $\langle q_i \rangle$ as time goes on as predicted by equation (21).

The degree assortativity coefficient plays an important role in the model. By random degree-preserving rewiring of the original BA network, we tune its degree assortativity to $r \approx 0.3$ and $r \approx -0.2$, respectively [49]. As shown in figure 3 we find that in this case the mean values of q_i still tend toward the mean values of $\langle q_i \rangle$, though strictly speaking the convergence in equations (20) and (21) no longer holds for strongly correlated networks, assortative or disassortative. The tendency slows down with the increase of r, which is intuitively reasonable. For networks with positive degree correlation, the agents with large degrees are apt to link with those with similar degrees. In figure 3 a specific opinion is assigned to the largest degree agents; if the agents are selected to update their opinions, due to the positive correlation, it is very likely that their neighbors who are chosen still are the largest degree agents with the same opinion as theirs, leading to the slow tendency of $E(q_i)$ for assortative networks. However for $r < 0$ the opposite situation occurs resulting in the fast tendency of $E(q_i)$ to $E[\langle q_i \rangle]$.

The conventional wisdom is that social networks are positively assorted on degree and the biological and technological networks are negatively assorted. However recent research on the Internet community modifies the wide-spread belief. Online social networks show more diverse patterns, including disassortative, assortative and neutral mixing [44]. Thus according to figure 3, if most large-degree persons hold the same opinion, the opinion will diffuse faster in the disassortative social networks than in the assortative ones, whether the opinion is good or not.

For homogeneous networks, such as random graphs and small-world networks, due to the homogeneity of agent degrees, even though we could manually assign a specified opinion i to the largest degree agents, the initial $\langle q_i \rangle$ is still approximately equal to q_i. In this case the large-degree agents contribute a negligible weight to the $\langle q_i \rangle$.

Further we also study evolutions of mean values of q_i and $\langle q_i \rangle$ for randomly distributed u following specific probability densities. Figure 4 shows the simulation results for three different distributions: exponential, power-law and uniform distributions. We find that the mean values of q_i still tend toward those of $\langle q_i \rangle$ over time. However, strictly speaking, in this case the convergence in equations (20) and (21) does not hold since u is a random variable.

3.2. When v is degree dependent

In social life, the assumption that each agent has the same v is not so realistic. For practical purposes different agents with different degrees (corresponding to social status and prestige) can have disparate v. If v is degree dependent, the evolution scenario of opinions will change remarkably. In this case the original v is denoted by v_k, and from equation (13) we have

$$\mathrm{d}q_{i,k} = \left(\langle q_i \rangle - q_{i,k}\right) v_k \,\mathrm{d}t + \frac{1}{\sqrt{n_k}} \sigma_{i,k} \sqrt{v_k} \,\mathrm{d}B_t, \tag{22}$$

thus

$$\mathrm{d}\langle q_i(t) \rangle = \frac{\sum_k k p_k [(\langle q_i \rangle - q_{i,k}) v_k \,\mathrm{d}t]}{\sum_k k p_k} + \frac{\sum_k k p_k \left[\frac{1}{\sqrt{n_k}} \sigma_{i,k} \sqrt{v_k} \,\mathrm{d}B_t\right]}{\sum_k k p_k}. \tag{23}$$

$\langle q_i(t) \rangle$ is no longer a martingale.

When the number N of agents is very large, we have

$$q_{i,k}(t) = q_{i,k}(0)\, \mathrm{e}^{-v_k t} + \langle q_i \rangle (1 - \mathrm{e}^{-v_k t}) \tag{24}$$

and equation (18) still holds.

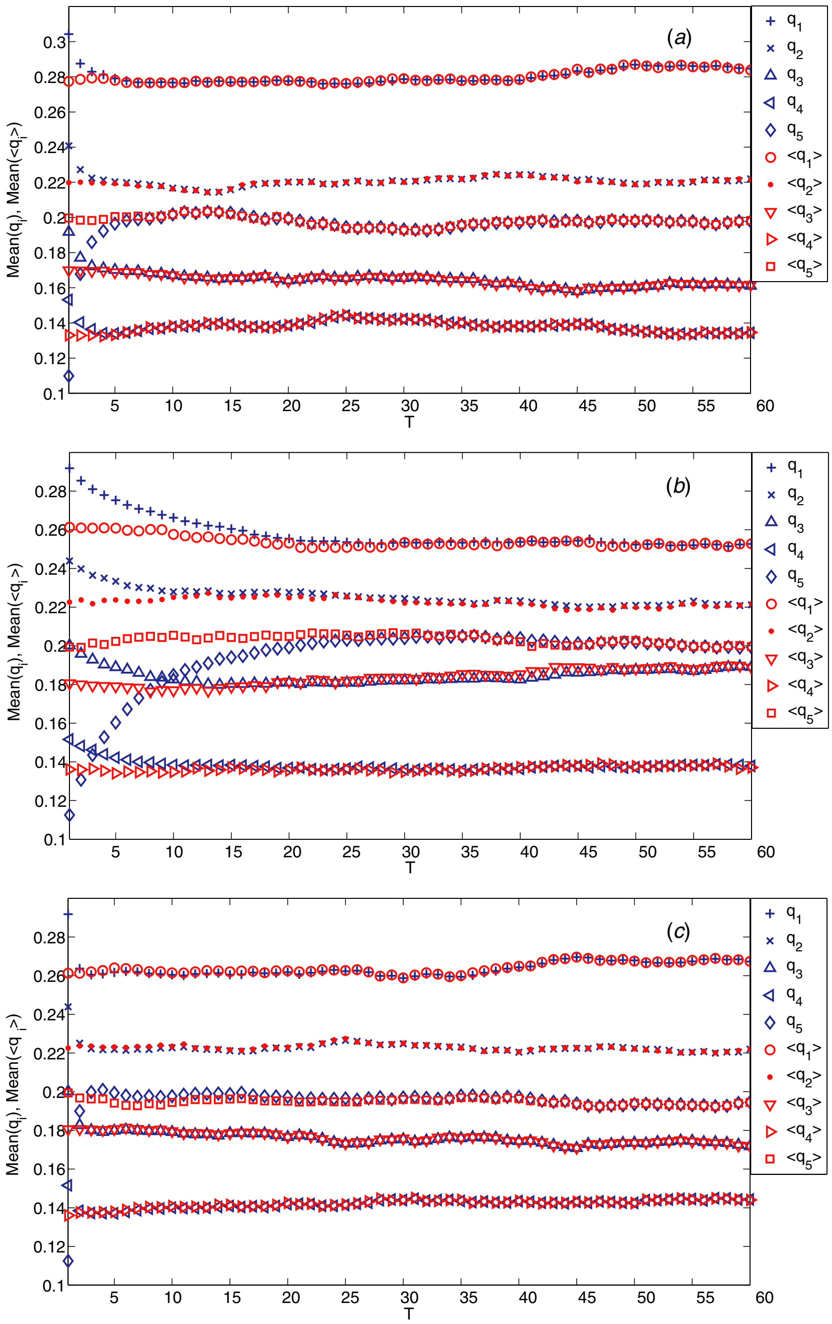

Figure 3. Evolutions of mean values of q_i and $\langle q_i \rangle$ for $v = 0.7$. The unit of time T is N/v steps. The opinion of the 100 agents with the largest degrees is assigned to be 5. The simulation is performed on a BA network with 10^4 agents and mean degree 6. (a) Degree assortativity coefficient $r \approx 0$, (b) $r \approx 0.3$ and (c) $r \approx -0.2$.

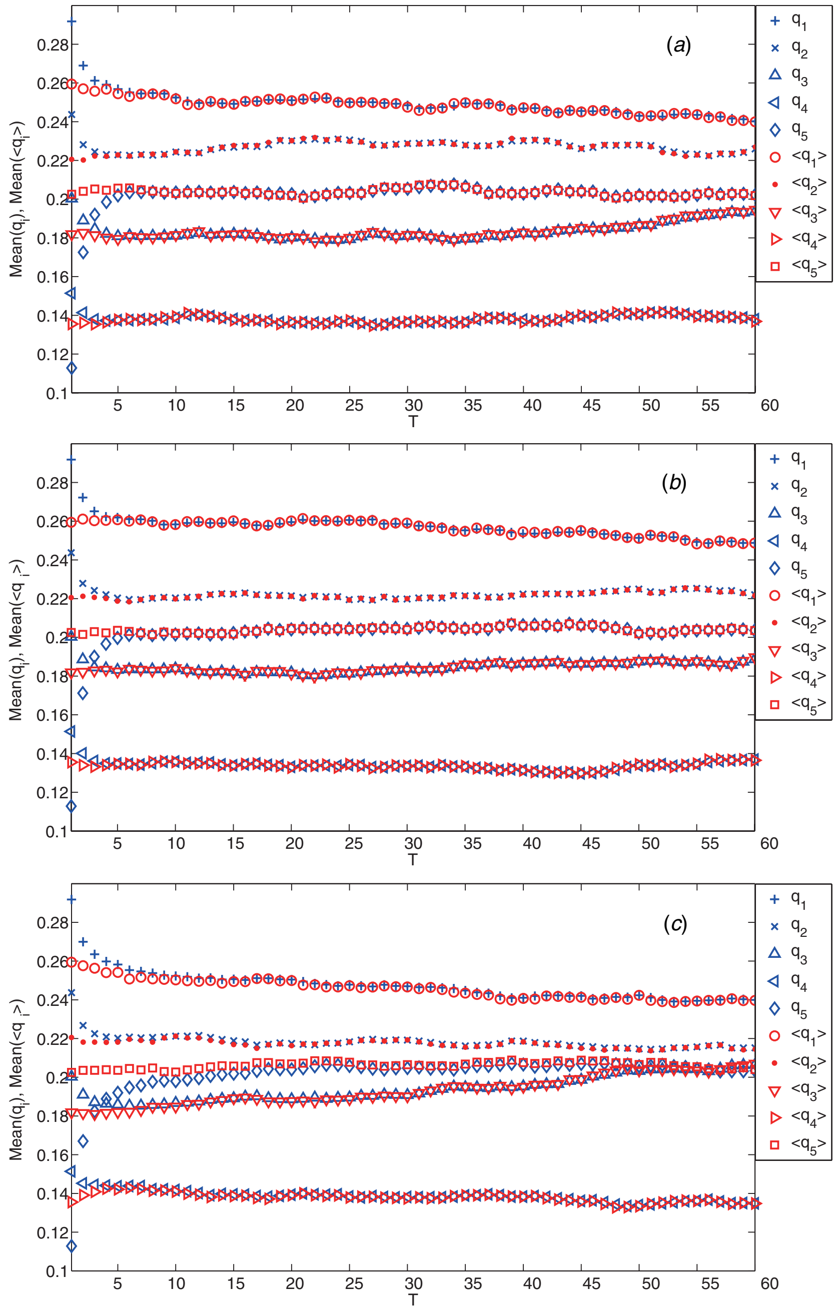

Figure 4. Evolutions of mean values of q_i and $\langle q_i \rangle$ for randomly distributed u following specific probability densities: (a) exponential distribution in (0,1) with mean 0.1; (b) power-law distribution $p_u = 5 \times 10^{-5} \cdot u^{-2.5}$ in (0,1); (c) uniform distribution in (0,1). The unit of time T is $N/\langle v \rangle$ steps. The opinion of the 100 agents with the largest degrees is assigned to be 5. The simulation is performed on a BA network with 10^4 agents and mean degree 6.

Further, we can obtain

$$q_i(t) = \sum_k n_k q_{i,k}(0)\, \mathrm{e}^{-\upsilon_k t} \Big/ N + \langle q_i \rangle \left(1 - \sum_k n_k\, \mathrm{e}^{-\upsilon_k t} \Big/ N\right), \tag{25}$$

thus equations (20) and (21) also still hold.

When υ_k is degree dependent, $\langle q_i \rangle$ does not possess the martingale property, however $E(q_i)$ still converges to $E[\langle q_i \rangle]$ as time goes on. From equations (23) and (18), for a large population we find

$$\lim_{t\to\infty} \mathrm{d}\langle q_i(t)\rangle = \frac{\sum_k k p_k \left[\frac{1}{\sqrt{n_k}} \sigma_{i,k} \sqrt{\upsilon_k}\, \mathrm{d}B_t\right]}{\sum_k k p_k}, \tag{26}$$

and in the long time limit the martingale feature of $\langle q_i \rangle$ can be recovered.

In real life the people with high social status could possess more capability of judgment and self-confidence. They may be more likely to insist on their own opinions, not subject to the opinions of their acquaintances. Thus the larger the agent's degree, the more possible that the agent will keep his current opinion, i.e. υ_k is a decreasing function of k. Specifically we may suppose that υ_k decays algebraically with k according to a power law $\upsilon_k = ck^{-\alpha}$, where $0 < c \leqslant 1$ and $\alpha \geqslant 0$ are two constants. When $\alpha = 0$, $\upsilon_k = c$ and the case that υ is degree independent is recovered. Especially when $\alpha = 1$, from equation (23) we have

$$\mathrm{d}\langle q_i(t)\rangle = \frac{c[\langle q_i \rangle - q_i]\mathrm{d}t}{\langle k \rangle} + \frac{\sum_k k p_k \left[\frac{1}{\sqrt{n_k}} \sigma_{i,k} \sqrt{\upsilon_k}\, \mathrm{d}B_t\right]}{\sum_k k p_k}. \tag{27}$$

Thus in the limit of long time when $q_i(t)$ converges to $\langle q_i \rangle$, equation (27) becomes equation (26). Figure 5 shows the simulation results performed on the BA network for $\upsilon_k = 0.9 \cdot k^{-1}$ which validate equation (21).

It is noteworthy that our comment that the people with high social status could possess more self-confidence is only an assumption. To the best of our knowledge there is no sociological study corroborating it. To obtain some information on the distribution of u or its correlation with degree in population, one would have to make a detailed social survey of floating voters. Besides, all error bars behave similarly in figures 2–5 where all agents are undecided, and the standard deviations of q_i and $\langle q_i \rangle$ increase with T.

4. Evolution of opinions for decided and undecided agents

We now further extend our model to allow a fraction of agents to have fixed opinions which do not change over time ($u = 1$). The stubborn agents can affect others but they themselves cannot be affected by others. The stubbornness of agents may have more general real-life meaning, namely a person or party (represented still by one node) that through political propaganda influences all its neighbors without getting significative opinion feedback from them. In a social context, the obstinate agents can be regarded as decided persons while the other agents as undecided people.

Let $s_{i,k}$ be the fraction of agents holding opinion i forever in all degree-k agents. In this case equation (4) can be modified as

$$\begin{cases} P_{\bar{i}\to i}(k) = p_k\left(1 - q_{i,k} - \sum_{j\neq i} s_{j,k}\right)\upsilon_k \langle q_i \rangle \\ P_{i\to \bar{i}}(k) = p_k(q_{i,k} - s_{i,k})\upsilon_k\,(1 - \langle q_i \rangle)\,. \end{cases} \tag{28}$$

Repeating the steps in the previous section, we obtain

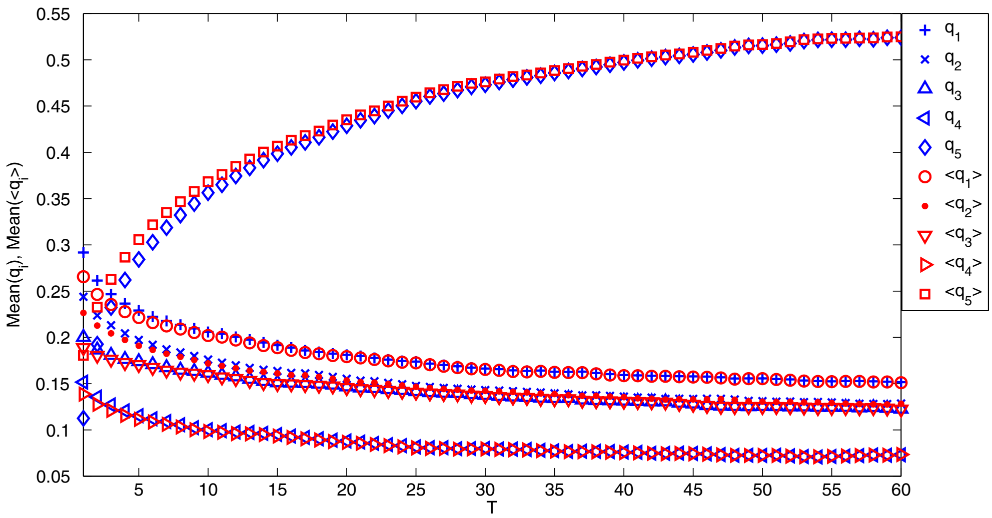

Figure 5. Evolutions of mean values of q_i and $\langle q_i \rangle$ for $v_k = 0.9 \cdot k^{-1}$. The unit of time T is $N/v_{\langle k \rangle}$ steps. The opinion of the 100 agents with the largest degrees is assigned to be 5. The simulation is performed on a BA network with 10^4 agents and mean degree 6.

$$\begin{aligned} \mathrm{d}q_{i,k} &= \left[\langle q_i \rangle - q_{i,k} + s_{i,k} \left(1 - \langle q_i \rangle\right) - \sum_{j \neq i} s_{j,k} \cdot \langle q_i \rangle \right] v_k \, \mathrm{d}t + \frac{1}{\sqrt{n_k}} \sigma'_{i,k} \sqrt{v_k} \, \mathrm{d}B_t \\ &= \left[\left(1 - \sum_j s_{j,k}\right) \langle q_i \rangle - q_{i,k} + s_{i,k} \right] v_k \, \mathrm{d}t + \frac{1}{\sqrt{n_k}} \sigma'_{i,k} \sqrt{v_k} \, \mathrm{d}B_t. \end{aligned} \tag{29}$$

For a large population, applying the mean-field approximation, we have

$$q_{i,k}(t) = q_{i,k}(0)\, \mathrm{e}^{-v_k t} + \left[\left(1 - \sum_j s_{j,k}\right) \langle q_i \rangle + s_{i,k} \right] (1 - \mathrm{e}^{-v_k t}), \tag{30}$$

and thus

$$\lim_{t \to \infty} q_{i,k}(t) = \left(1 - \sum_j s_{j,k}\right) \langle q_i \rangle + s_{i,k}. \tag{31}$$

When $s_{i,k} = 0$ the above formula is reduced to equation (18). From equation (30) we obtain

$$\begin{aligned} q_i(t) = {} & \sum_k n_k q_{i,k}(0)\, \mathrm{e}^{-v_k t} \Big/ N + \langle q_i \rangle - \langle q_i \rangle \left(\sum_k n_k \, \mathrm{e}^{-v_k t} \right) \Big/ N \\ & - \langle q_i \rangle \left[\sum_k n_k \cdot \left(\sum_j s_{j,k} \right) \right] \Big/ N \\ & + \langle q_i \rangle \left[\sum_k n_k \cdot \left(\sum_j s_{j,k} \cdot \mathrm{e}^{-v_k t} \right) \right] \Big/ N + s_i - \sum_k n_k s_{i,k}\, \mathrm{e}^{-v_k t} \Big/ N, \end{aligned} \tag{32}$$

where s_i represents the fraction of decided agents holding opinion i forever. Thus

$$\lim_{t\to\infty} q_i(t) = \langle q_i\rangle - \langle q_i\rangle\left[\sum n_k \cdot \left(\sum_j s_{j,k}\right)\right]\bigg/ N + s_i,$$
$$= (1-s)\langle q_i\rangle + s_i, \tag{33}$$

where s represents the fraction of decided agents. When $s_i = 0$ the above formula is reduced to equation (20).

When v_k is a degree-independent constant, from equation (29), we have

$$\frac{\mathrm{d}\langle q_i\rangle}{\mathrm{d}t} = \left[\langle s_i\rangle - \langle q_i\rangle \sum_j \langle s_j\rangle\right] v, \tag{34}$$

and the equilibrium condition is obtained by

$$\left[\langle s_i\rangle - \langle \hat{q}_i\rangle \sum_j \langle s_j\rangle\right] v = 0, \tag{35}$$

where $\langle \hat{q}_i\rangle = \lim_{t\to\infty}\langle q_i(t)\rangle$. Thus

$$\langle \hat{q}_i\rangle = \langle s_i\rangle \bigg/ \sum_j \langle s_j\rangle. \tag{36}$$

From equations (33) and (36) we find that, as v_k is degree independent and $t\to\infty$, both $\langle q_i\rangle$ and q_i converge to fixed proportions which only depend on the distribution of decided agents and are independent of the initial assignment of undecided agents. In particular if all the decided persons stick to the same opinion i, $\langle \hat{q}_i\rangle = 1 = \hat{q}_i$, i.e. opinion i will finally pervade the whole population no matter how small the number of stubborn people is. That seems to validate the proverb ‘Success belongs to the persevering’. Again the weighted fractions in equation (36) demonstrate that high-degree agents are more influential to the final state of the dynamical process. The scenario in the end is the equilibrium state with the coexistence of diverse opinions that are just the ones held by the decided people, although it is possible that the convergence needs such a long time that it can never be reached in reality. Real social networks always evolve over time, thus the convergence also will depend on their evolution. There are two timescales, one from the opinion convergence and the other from the networks. Further if the decided agents are randomly distributed, $\langle \hat{q}_i\rangle = \langle s_i\rangle \big/ \sum_j \langle s_j\rangle = s_i/s = \hat{q}_i$.

Figure 6 shows the simulation results performed on the BA network. The initial agents both with opinions 1 and 2 are the mixing of decided and undecided ones, while all the other agents holding opinions 3, 4 and 5 are undecided ones. Besides we assume that the 100 largest degree agents are the stubborn ones with opinion 1; $s_1 = 0.3$, $s_2 = 0.2$ and $s = 0.5$. According to equations (33) and (36), and the initial distribution of decided agents, we obtain $\hat{q}_1 = 0.63667$, $\langle \hat{q}_1\rangle = 0.67335$, $\hat{q}_2 = 0.36333$, $\langle \hat{q}_2\rangle = 0.32665$ and $\langle \hat{q}_i\rangle = \hat{q}_i = 0$ for $i = 3$, 4 and 5. As shown in figure 6, in this case the standard deviations of $\langle q_i\rangle$ and q_i are all very small and both $\langle q_i\rangle$ and q_i fluctuate slightly around their limits as T is sufficiently large, which can result from the stubborn agents in networks. Besides the average values of q_i and $\langle q_i\rangle$ are in good agreement with the limits $\hat{q}_i$ and $\langle \hat{q}_i\rangle$ respectively for large T.

5. Conclusions and discussions

As we know unanimity is one of the most important aspects of social group dynamics. Everyday life presents many situations in which it is necessary for a group to reach shared

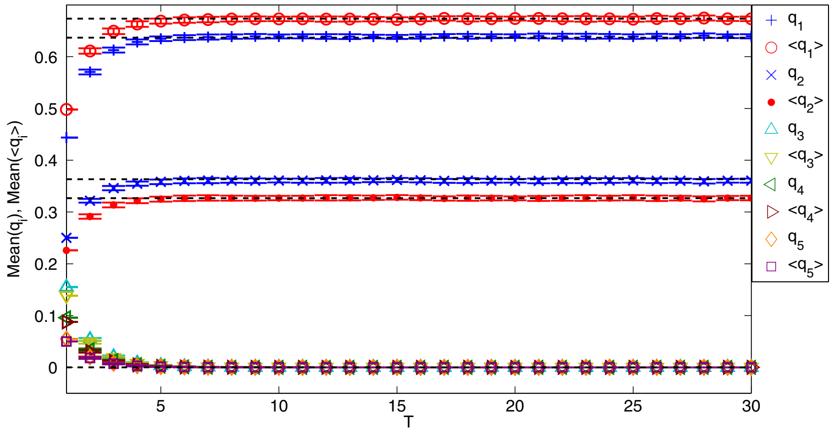

Figure 6. Evolutions of mean values of q_i and $\langle q_i \rangle$ for $v = 0.8$. Error bars with ± 1 standard deviation are shown. The unit of time T is $N/[v(1-s)]$ steps. The 100 largest degree agents are assumed to be the stubborn ones with opinion 1. The simulation is performed on a BA network with 10^4 agents and mean degree 6. The dashed lines indicate the limit values for q_i and $\langle q_i \rangle$ based on equations (33) and (36).

decisions. Consensus makes some opinion stronger and amplifies its impact on society. However in real life, after a long-term interaction of people in a society, the phenomenon of coexistence of different opinions can also exist, which embodies the strong heterogeneity of individual character.

In the paper, we study a model of discrete opinion dynamics based on social influence. If all persons are undecided ones in a network, we find that for any number of initial opinions, when the self-confidence parameter is a constant, the weighted proportion of each opinion is a martingale. The fraction of each opinion will gradually converge to its weighted proportion and the tendency can slow down with the increase of degree assortativity coefficient, whether the self-confidence level is degree dependent or not. Further we also consider the scenario that in the population there are stubborn persons whose opinions never change and find that for constant self-confidence level, both fraction and weighted proportion of each opinion will converge to fixed values which only depend on the distribution of stubborn persons, and are independent of the initial assignment of undecided persons.

In the model, if all agents are undecided ones, the different opinions of individuals will reach consensus in the end for a finite population. However if there are decided persons in a society, the state of coexistence of diverse opinions can occur. In both cases we quantitatively show the influence of agents with large degrees on the final state of opinion evolution.

We can naturally extend the model to the condition that the social influence is not symmetrical. For instance, agent A can affect the decisions or choices of agent B; however, the opposite case almost never occurs. This corresponds to the model on directed networks in which the fact that A points to B means A can influence B but not the other way round. Evidently if we replace the degree by out-degree in the above text, the previous derived conclusions still hold.

Opinion dynamics is similar in some ways to information spreading in networks where information is propagated from the persons who know it to those who do not know it [46, 50], if we regard opinion as a kind of special information. Information could propagate to the whole networks and some specific opinion could also prevail in the whole population. However, in opinion dynamics there exist several different opinions which initially are distributed in the networks and different opinions belonging to two connected agents could interact with each other, while in information spreading there usually only exists a kind of information (e.g. confidential or risky information) diffusing in networks.

The parameters studied in this paper are all global measures (r, p_k, etc). It is clear that for the same values of global network measures, the connections between persons may be distributed in many completely different ways. Recent research shows that for spreading information in networks, the different ways of connection could change the accessibility of information from one node to another [47, 51].

In our model all agents update their opinions asynchronously and independently of each other. In future work we can study spontaneous changes of opinions in an improved model. Besides our model is based on unweighted networks, which is not realistic to some extent. Certainly in social life, changing opinion strongly depends on how close the acquaintance is between the two interacting persons. Good friends have a stronger influence on each other than not such good friends [52]. For the discrete opinion dynamics on weighted networks, good friends connected by large-weight edges can interact more frequently than ordinary friends connected by low-weight edges, or the v value is weight dependent and a person is more likely to follow the suggestions of his intimate friends than general friends. Opinion dynamics on weighted networks will be our research focus in our future work.

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